

The renormalization of entanglement in the anisotropic Heisenberg (XXZ) model

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We have applied our recent approach (Kargarian, *et.al* Phys. Rev. A **76**, 60304 (R) (2007)) to study the quantum information properties of the anisotropic $s=1/2$ Heisenberg chain. We have investigated the underlying quantum information properties like the evolution of concurrence, entanglement entropy, nonanalytic behaviours and the scaling close to the quantum critical point of the model. Both the concurrence and the entanglement entropy develop two saturated values after enough iterations of the renormalization of coupling constants. This values are associated with the two different phases, i.e Néel and spin liquid phases. The nonanalytic behaviour comes from the divergence of the first derivative of both measures of entanglement as the size of system becomes large. The renormalization scheme demonstrates how the minimum value of the first derivative and its position scales with an exponent of the system size. It is shown that this exponent is directly related to the critical properties of the model, i.e. the exponent governing the divergence of the correlation length close to the quantum critical point. We also use a renormalization method based on the quantum group concept in order to get more insight about the critical properties of the model and the renormalization of entanglement.

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I. INTRODUCTION

The quantum information theory (QIT) and condensed matter physics share in using the entanglement¹. The main motivation behind such interest is two folds: On the one hand in QIT the entanglement is recognized as an essential resource for implementing some quantum information tasks such as quantum computations, quantum cryptography and densecoding². It is also believed that the protocol based on the quantum entangled state has much speed than the classical ones. On the other hand entanglement is a unique measure of the quantum correlation of a pure state in condensed matter physics. Thus, for condensed matter systems the entanglement can be considered as some kind of correlation which appears in the pure states.

The role of entanglement in quantum phase transition (QPT)³ is of considerable interest⁴. Quantum phase transitions occur at absolute zero temperature. They are induced by the change of an external parameter or coupling constant, and are driven by quantum fluctuations. Quantum as well as classical phase transitions are characterized by detecting nonanalytic behaviors in some physical properties of the system. It is often accompanied by a divergence in some correlation functions, but quantum systems possess additional correlations which do not exist in a classical counterparts, *the entanglement*. Entanglement is a direct measure of quantum correlations and shows nonanalytic behavior such as discontinuity in the vicinity of the quantum critical point^{5,6}. In the past few years the subject of several activities were to investigate the behavior of entanglement in the vicinity of quantum critical point for different spin models^{4,7,8,9,10,11} as well as itinerant systems^{12,13,14}.

In our recent work¹⁵ we have introduced the notion of

renormalization of entanglement and showed that this notion truly describe the nonanalytic behavior of the derivative of entanglement close to critical point of the Ising model in Transverse Field (ITF). Moreover, we have investigated how the critical point is reached by increasing the size of system via the renormalization group (RG) approach. The finite size scaling demonstrates that the RG of entanglement truly captures the critical behavior of the model. The renormalization of quantum states has also been introduced in terms of matrix product states¹⁶.

In this work we have applied the same approach to the anisotropic spin $1/2$ Heisenberg (XXZ) model in order to investigate the critical behavior of the model using the evolution of entanglement through the quantum renormalization group^{17,18} close to quantum critical point. In this model the concurrence between the two nearest-neighbor sites develops a maximum at the isotropic point ($\Delta = 1$) without any singularity in its first derivative¹⁹, i.e it vanishes at the critical point $\Delta = 1$. However, in our approach a singularity, i.e the divergence of first derivative, is seen at the critical point. It is expected that our approach is efficient and powerful to study the quantum phase transition in various systems since the thermodynamic limit of the model is simply captured through an analytic RG equations. Quantum renormalization group with fixed boundary conditions give rise only to a non-trivial fixed point $\Delta = 1$ and trivial fixed points $\Delta = 0, \infty$ for the XXZ model while for every point in the $0 \leq \Delta \leq 1$ region the model is critical, i.e gapless. A modification of the boundary conditions by means of quantum group concept²⁰ restores the gapless properties of $0 \leq \Delta \leq 1$, although the critical behavior of the model and its scaling properties remain unchanged from the gapped phase.

This article is organized as follows. In section-II the main idea of quantum renormalization group is briefly re-

viewed and in section-III this idea is applied to the XXZ model. Sections-IV discusses the main idea of this paper about the renormalization of entanglement entropy and concurrence is introduced and the section-V is devoted to the quantum group analysis in order to get more insights about the critical features of the model. Finally in section-VI we summarize our results.

II. QUANTUM RENORMALIZATION GROUP

The main idea of the RG method is the mode elimination or thinning of the degrees of freedom followed by an iteration which reduces the number of variables step by step until reaching a fixed point. In Kadanoff's approach, the lattice is divided into blocks. Each block is treated independently to build the projection operator onto the lower energy subspace. The projection of the inter-block interaction is mapped to an effective Hamiltonian (H^{eff}) which acts on the renormalized subspace^{21,22}. The procedure starts by decomposing the Hamiltonian into two parts,

$$H = H^B + H^{BB}, \quad (1)$$

where the block Hamiltonian H^B is a sum of commuting terms, each acting on different blocks. The inter-block interaction is represented by H^{BB} . The perturbative implementation of this method has been discussed in Refs.[21,24]. We will shortly present this approach in the 1st order correction. The zeroth order effective Hamiltonian is given by

$$H_0^{eff} = P_0 H^B P_0, \quad (2)$$

where P_0 is a projection operator. Since H^B is a sum of disconnected block Hamiltonians

$$H^B = \sum_{I=1}^{N'} h_I^B,$$

one can search for a solution of P_0 in a factorised form

$$P_0 = \prod_{I=1}^{N'} P_0^I,$$

where N' is the number of blocks. In the standard quantum renormalization group, P_0^I is given by

$$P_0^I = \sum_{i=1}^k |\psi_i\rangle\langle\psi_i|,$$

where $|\psi_i\rangle$ ($i = 1, \dots, k$) are the k lowest energy eigenstates of h_I^B . The interaction between blocks define the 1st order correction by the following equation

$$H_1^{eff} = P_0 H^{BB} P_0. \quad (3)$$

The effective (renormalized) Hamiltonian is then

$$H^{eff} = H_0^{eff} + H_1^{eff}. \quad (4)$$

We will implement this approach in the next sections to obtain the quantum properties of the XXZ spin chain.

III. RENORMALIZATION OF THE XXZ MODEL

The Hamiltonian of XXZ model on a periodic chain of N sites is

$$H(J, \Delta) = \frac{J}{4} \left(\sum_i^N \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right) \quad (5)$$

where $J, \Delta > 0$, J is the exchange coupling, Δ is the anisotropy parameter and σ^α ($\alpha = x, y, z$) are Pauli matrices. For $\Delta = 1$, the Hamiltonian is $SU(2)$ symmetry invariant, but for $\Delta \neq 1$ the $SU(2)$ symmetry breaks down to the $U(1)$ rotational symmetry around the z -axis. The model is exactly solvable by means of Bethe Ansatz as far as the rotational symmetry exists. It is known that for $0 \leq \Delta \leq 1$ the model is gapless with quasi-long range ordered where the correlations decay algebraic with no magnetic long range order. For $\Delta > 1$, the symmetry is reduced to Z_2 and a gap opens which is in the universality class of one dimensional antiferromagnetic Ising chain. Indeed the third term in the Hamiltonian causes ordering in the system and as Δ tends to infinity the Néel state is the dominant phase of the system. The first two terms in the Hamiltonian extend the quantum fluctuations in the system and result in the corruption of the Néel ordering. It is shown that the competition between the quantum fluctuations and ordering yield a maximum value of concurrence between the two nearest neighbor sites which show a scaling behavior at the critical point $\Delta = 1^{25, 26}$.

To implement the idea of QRG to calculate the entanglement and concurrence we use a three site block procedure where the block Hamiltonian is

$$h_I^B = \frac{J}{4} \left[(\sigma_{1,I}^x \sigma_{2,I}^x + \sigma_{2,I}^x \sigma_{3,I}^x + \sigma_{1,I}^y \sigma_{2,I}^y + \sigma_{2,I}^y \sigma_{3,I}^y) + \Delta (\sigma_{1,I}^z \sigma_{2,I}^z + \sigma_{2,I}^z \sigma_{3,I}^z) \right]. \quad (6)$$

In this case the inter-block (H^{BB}) and intra-block (H^B) Hamiltonians are

$$H^{BB} = \frac{J}{4} \left[\sum_I^{N/3} (\sigma_{3,I}^x \sigma_{1,I+1}^x + \sigma_{3,I}^y \sigma_{1,I+1}^y + \Delta \sigma_{3,I}^z \sigma_{1,I+1}^z) \right], \quad (7)$$

$$H^B = \sum_I^{N/3} h_I^B. \quad (8)$$

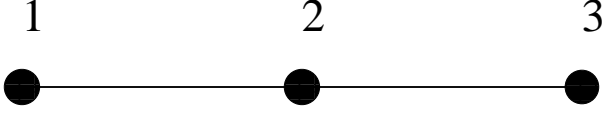


FIG. 1: Three sites in a block labeled sequentially.

A remark is in order here, choosing the three site block is essential here to get a self similar Hamiltonian after each RG step. An odd site XXZ Hamiltonian has two degenerate ground states which are used to construct the projection operator of quantum RG (QRG). These degenerate ground states are

$$|\phi_0\rangle = \frac{1}{\sqrt{2+q^2}}(|\uparrow\uparrow\downarrow\rangle + q|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle), \quad (9)$$

$$|\phi'_0\rangle = \frac{1}{\sqrt{2+q^2}}(|\uparrow\downarrow\downarrow\rangle + q|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle), \quad (10)$$

where q is

$$q = \frac{-1}{2}[\Delta + \sqrt{\Delta^2 + 8}]. \quad (11)$$

The corresponding energy is

$$e_0 = -\frac{J}{4}[\Delta + \sqrt{\Delta^2 + 8}], \quad (12)$$

and $|\uparrow\rangle, |\downarrow\rangle$ are the eigenstates of σ^z .

The projection operator (P_0) for the I -th block is defined

$$P_0^I = |\uparrow\rangle_I \langle\phi_0| + |\downarrow\rangle_I \langle\phi'_0|, \quad (13)$$

where $|\uparrow\rangle_I$ and $|\downarrow\rangle_I$ are renamed states of each block to represent the effective site degrees of freedom. The renormalization of Pauli matrices are given by

$$P_0^I \sigma_{i,I}^\alpha P_0^I = \xi_i^\alpha \sigma'^\alpha_I \quad (i = 1, 2, 3 ; \alpha = x, y, z), \quad (14)$$

where

$$\begin{aligned} \xi_1^{x,y} = \xi_3^{x,y} &= \frac{2q}{2+q^2} \quad , \quad \xi_1^z = \xi_3^z = \frac{q^2}{2+q^2} \\ \xi_2^{x,y} &= \frac{2}{2+q^2} \quad , \quad \xi_2^z = \frac{2-q^2}{2+q^2}, \end{aligned} \quad (15)$$

where the indices 1, 2 and 3 refer to sites labeled in a single block based on Fig.(1).

The effective Hamiltonian (using Eq.(4)) of the renormalized chain is again an XXZ chain with the scaled couplings

$$H^{eff} = \frac{J'}{4} \left[\sum_i^{N/3} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \Delta' (\sigma_i^z \sigma_{i+1}^z) \right], \quad (16)$$

where

$$J' = J \left(\frac{2q}{2+q^2} \right)^2 \quad , \quad \Delta' = \Delta \frac{q^2}{4}. \quad (17)$$

The stable and unstable fixed points of the QRG equations is obtained by solving $\Delta' = \Delta \equiv \Delta^*$. The stable fixed points locate at zero and infinity while $\Delta^* = 1$ stands for the unstable fixed point which specify the critical point of the model. Starting with any initial value for $\Delta > 1$, the coupling constant flows toward infinity showing that the model falls into the universality class of Ising model, while for $\Delta < 1$ the stable $\Delta = 0$ fixed point is touched. As we have mentioned previously, for $0 \leq \Delta \leq 1$ the model represents a spin fluid phase. The transition between the two phases are truly captured by real space QRG. The main discrepancy of our results on the Ising model in transverse field¹⁵ (ITF) and the present one on XXZ comes from the fact that XXZ model is critical for all values of $0 \leq \Delta \leq 1$ but QRG equations do not to show the whole critical region except $\Delta = 0$. In fact the QRG prescription only represents the masslessness property of the XY fixed point ($\Delta = 0$). However, when the coarse graining procedure implemented by some appropriate boundary conditions in order to get more correlation between blocks as is done in the quantum group method²⁰, the critical line of the model is truly predicted (see section V).

IV. RENORMALIZED ENTANGLEMENT AND ITS SCALING PROPERTY

A. Entanglement and concurrence

In this subsection we calculate the concurrence and entanglement of the XXZ chain using our proposal which implements the idea of renormalization group. As we have mentioned previously, a finite size block is treated exactly to calculate the physical quantities. The coupling constants of a finite block are renormalized via the QRG prescription to give the large size behaviour. The XXZ spin 1/2 Hamiltonian has two degenerate ground state in the three site block, however to define the density matrix we have to consider one of them. Thus, the density matrix is defined by

$$\varrho = |\phi_0\rangle \langle\phi_0|, \quad (18)$$

where $|\phi_0\rangle$ has been introduced in Eq.(9). The results will be the same if we consider $|\phi'_0\rangle$ to construct the density matrix.

There are basically two choices to define the concurrence and entanglement for a three site block. (1) The symmetric case, in which the concurrence between site 1 and 3 is obtained by summing over the degrees of freedom of site 2. And the entanglement between site 2 and remaining sites of the block as measured by von-neuman entropy. (2) We sum over site 1 or 3 and get the concurrence between the middle site (2) and one site at the

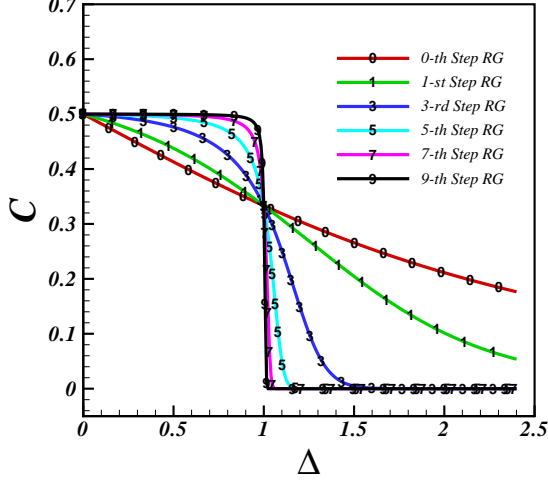


FIG. 2: (color online) Representation of the evolution of concurrence in terms of RG iterations (steps).

corner side of the block (see Fig.(1)). Without loss of generality we only concentrate on case (1). The density matrix defined in Eq.(18) is traced over site 2 degrees of freedom to get the reduced density matrix for site 1 and 3 ($\varrho_{1,3}$) which gives

$$\varrho_{13} = \frac{1}{2+q^2} \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (19)$$

The corresponding eigenvalues of $\hat{R} = \varrho_{13}\varrho_{13}^*$ (where $\tilde{\varrho}_{13} = (\sigma_1^y \otimes \sigma_3^y)\varrho_{13}^*(\sigma_1^y \otimes \sigma_3^y)$) are in ascending order

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = \frac{4}{(2+q^2)^2}. \quad (20)$$

Thus, the concurrence is obtained

$$C_{13} = \max\{\lambda_4^{1/2}, 0\} = \frac{2}{2+q^2}. \quad (21)$$

According to Eq.(11), C_{13} is a function of Δ . The renormalization of Δ defines the evolution of concurrence as the size of system becomes large. We have plotted in FIG.2 the value of C_{13} versus Δ for different QRG iterations.

The plots of C_{13} versus Δ for different QRG steps cross each other at the scale invariant critical point, $\Delta_c \equiv \Delta^* = 1$. By increasing the size of system (higher QRG steps) the concurrence develops two different behaviours which are separated at Δ_c . After enough QRG steps the value of C_{13} is saturated for $0 \leq \Delta < 1$ which shows the existence of quantum correlations between two blocks at large distances in an infinite chain that is effectively described by a three site model with the renormalized coupling constants. In this region ($0 \leq \Delta \leq 1$) the

quantum fluctuations arising from the transverse interactions have dominant effect and destroy any long range order. In spite of lacking a long range order, the evolution of concurrence via RG indicates that the spin fluid phase contains the quantum correlations, i.e the qubits in the presence of quantum fluctuations are quantum correlated. For $\Delta > 1$, C_{13} saturates the *zero* value asymptotically to represent the lack of quantum correlation in the Ising limit. We have also examined the entanglement of formation²⁷ between sites 1 and 3 which can be obtained by the following relation:

$$E_1 = -y \log_2(y) - (1-y) \log_2(1-y), \\ y = \frac{1}{2} + \frac{1}{2} \sqrt{1 - C_{13}^2}. \quad (22)$$

The entanglement of formation shows similar behavior to FIG.2 because of monotonous relation between concurrence and entanglement of formation. In the symmetric case we can also study the entanglement between the middle site (2) with the remaining sites of the block. The amount of entanglement is given by von-Neuman entropy of reduced density matrix obtained after tracing out the remaining sites of block

$$\varrho_2 = \frac{1}{2+q^2} \begin{pmatrix} 2 & 0 \\ 0 & q^2 \end{pmatrix}. \quad (23)$$

The von-Neuman entropy which is the entanglement of site 2 is

$$E = -\frac{2}{2+q^2} \log_2 \frac{2}{2+q^2} - \frac{q^2}{2+q^2} \log_2 \frac{q^2}{2+q^2}. \quad (24)$$

The variation of entanglement (E) versus Δ has been plotted in Fig.3. Different plots show the evolution of E under QRG iterations. In other words, the different step of QRG show how the entanglement evolves when the size of chain is increased. Similar to C_{13} , E behaves as an order parameter which gets a nonzero value for $0 \leq \Delta < 1$ and zero for $\Delta > 1$ in the infinite size limit. *Nonzero* E for $0 \leq \Delta < 1$ verifies again that the state of model is entangled for $0 \leq \Delta < 1$ where the ground state is characterized by a gapless excitation and algebraic decay of spin correlations (the spin fluid phase). While, E is zero for the Néel state ($\Delta > 1$) which is not an entangled state.

As previously discussed the entanglement between the middle site (2) and the remaining ones tends to maximal entanglement for the region of coupling $0 \leq \Delta < 1$ after few RG steps Fig.3. While the entanglement of formation between sites 1 and 3 tends to a nonmaximal value, in fact this is a monogamy property of entanglement²⁸. On the basis of monogamy property, the maximal entanglement between two parties restricts their entanglement with third party and vis versa.

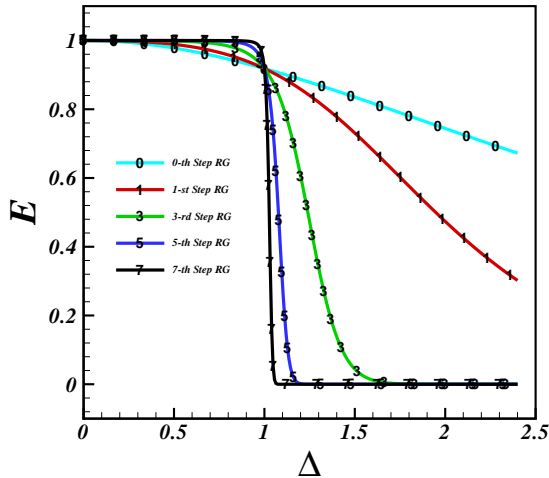


FIG. 3: (color online) The evolution of Entanglement entropy as the size of system grows (i.e. different RG iterations).

B. nonanalytic and scaling behavior

The non-analytic behavior in some physical quantity is a feature of second-order quantum phase transition. It is also accompanied by a scaling behavior since the correlation length diverges and there is no characteristic length scale in the system at the critical point. There have been a number of theoretical studies on the entanglement and quantum phase transitions. These studies show that the entanglement as a direct measure of quantum correlations indicate critical behavior such as diverging of its derivative as the phase transition is crossed. Osterloh, *et. al*⁴ have verified that the entanglement in the vicinity of critical point of ITF and XX model in transverse field show a scaling behavior. It has also been shown that the entanglement between a block of spins and the rest of system scales near the quantum critical point⁷. The scaling properties of entanglement with the size of block differs at and away from the critical point. At the critical point where the correlations decay algebraically entanglement scales logarithmically while it saturates away from the critical point²³. It can be interpreted in the framework of conformal field theory²⁹ associated with the quantum phase transition and the central charge of the theory. Indeed for the XXZ model, the entanglement entropy, i.e. the entanglement of a block (sublattice bipartition) with the rest of the system, demonstrates extremum behavior at the critical point³⁰.

In this work, we adopt a preferable and distinct way to study the block entanglement via the renormalization group approach. As we have stated in the RG approach for XXZ model, a large system, i.e. $N = 3^{n+1}$, can be effectively described by three sites with the renormalized couplings of the n -th RG iteration. Thus, the entanglement between the two renormalized sites represents

the entanglement between two parts of the system each containing $N/3$ sites effectively. In this respect we can speak of *block entanglement* - block-block entanglement or the entanglement between a block and the rest of system. It is shown that the first derivative of the entanglement shows a diverging behavior as the critical point is reached.

Having this in mind, the first derivative of concurrence is analyzed as a function of coupling Δ at different RG steps which manifest the size of system. The derivative of concurrence with respect to the coupling constant ($\frac{dF}{d\Delta}$) shows a singular behavior at the critical point. It is given by

$$\frac{dF^{(n)}}{d\Delta} = \frac{dF^{(n)}}{d\Delta_n} \frac{d\Delta_n}{d\Delta_{n-1}} \dots \frac{d\Delta_1}{d\Delta} \quad (25)$$

where $F^{(n)}$ stand for the concurrence or von-Neuman entropy and Δ_n the renormalized anisotropy coupling (Δ) at the n -th RG iteration. The singular behavior is the result of discontinuous change of C at $\Delta = \Delta_c$. We only concentrate on the nonanalytic behavior of entanglement between the middle site and remaining sites of each block. We have plotted $\frac{dE}{d\Delta}$ versus Δ in Fig.4 for different RG iterations which shows the singular behavior as the size of system becomes large (higher RG steps). A more detailed analysis shows that the position of the minimum (Δ_m) of $\frac{dE}{d\Delta}$ tends towards the critical point like $\Delta_m = \Delta_c + N^{-0.47}$ which has been plotted in Fig.5. Moreover, we have derived the scaling behavior of $y \equiv |\frac{dE}{d\Delta}|_{\Delta_m}$ versus N . This has been plotted in Fig.6 which shows a linear behavior of $\ln(y)$ versus $\ln(N)$. The exponent for this behavior is $|\frac{dE}{d\Delta}|_{\Delta_m} \sim N^{0.47}$. This results justify that the RG implementation of entanglement truly capture the critical behavior of the XXZ model at $\Delta = 1$.

Besides the entanglement entropy, two point quantum correlation as measured by concurrence develops a maximum at the critical point. The concurrence of the two nearest neighbor sites of a large chain has numerically been calculated and the maximum at the critical point is due to the counterbalance of ordering and quantum fluctuations¹⁹. In the Ising limit, i.e $\Delta \rightarrow \infty$, the Néel order is developed which makes the concurrence becomes zero. The same treatment also is seen in the renormalization of entanglement entropy or concurrence. As stated in section (2) the renormalisation equations develops two different phases for the XXZ model which are separated by the unstable fixed point $\Delta_c = 1$. For $\Delta > 1$ region which falls into the Ising universality class, both the entanglement entropy and concurrence (Fig.2) tends to zero as the iteration of RG equations increases, i.e the stable fixed point $\Delta \rightarrow \infty$ is reached.

A novel issue arises when the exponents of the scaling of Δ_m and the minimum value of von-Neuman entropy, as discussed in the previous paragraphs, are compared with the corresponding ones of the concurrence. The first derivative of concurrence tends to diverge at the quantum critical point. The position of minimum and the

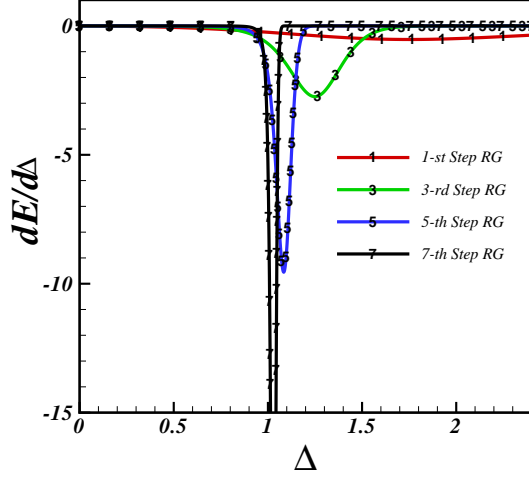


FIG. 4: (color online) First derivative of entanglement entropy and its manifestation towards diverging as the number of RG iterations (steps) increases (Fig.3).

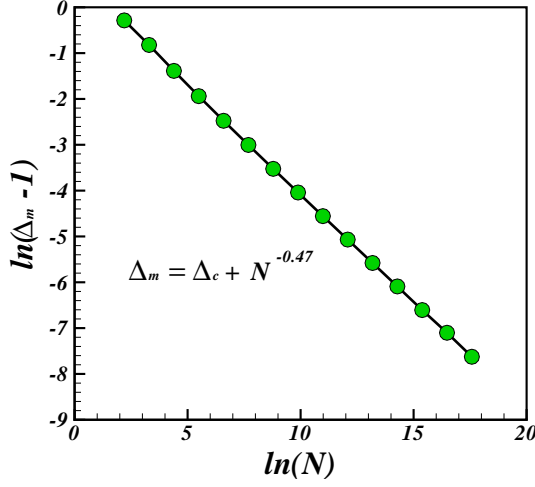


FIG. 5: (color online) The scaling behavior of Δ_m in terms of system size (N) where Δ_m is the position of minimum in Fig.4.

minimum value itself scale as $\Delta_m = \Delta_c + N^{-0.46}$ and $|\frac{dC}{d\Delta}|_{\Delta_m} \sim N^{0.48}$, respectively, as the size of the system becomes large. It is clear that the exponents are almost the same for the two measures of entanglement. It indicates that the entanglement entropy and concurrence scale in the same manner in the limit of large scale behavior and are associated to the long range behavior of the model close to the critical point. According to this intuitive picture of entanglement scaling and critical properties of the model we would like to emphasize

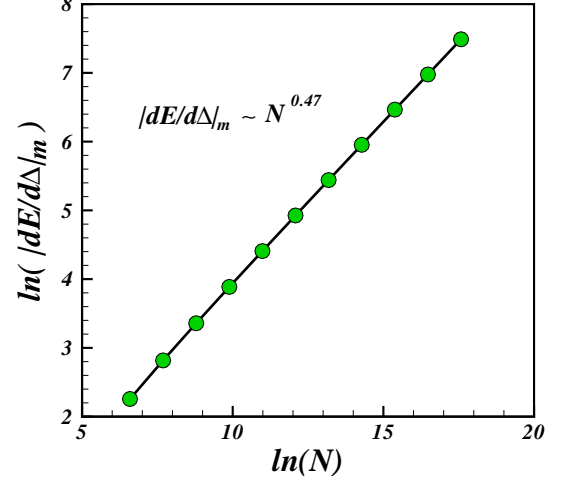


FIG. 6: (color online) The logarithm of the absolute value of minimum, $\ln(|dE/d\Delta|_{\Delta_m})$, versus the logarithm of chain size, $\ln(N)$, which is linear and shows a scaling behavior. Each point corresponds to the minimum value of a single plot of Fig.4.

that the exponent is directly related to the correlation length exponent close to the critical point. The correlation length exponent, ν , gives the behavior of correlation length in the vicinity of Δ_c

$$\xi \sim (\Delta - \Delta_c)^{-\nu} \quad (26)$$

Under the RG transformations (Eq.(17)) the correlation length scales as $\xi \rightarrow \xi^{(1)} = \xi/n_B$ where n_B stands for the number of sites in each block which is $n_B = 3$ in our procedure. So, it is easy to look for the n^{th} RG iteration

$$\xi^{(n)} \sim (\Delta_n - \Delta_c)^{-\nu} = \xi/n_B^n \quad (27)$$

which immediately leads to an expression for $|\frac{d\Delta_n}{d\Delta}|_{\Delta_c}$ in terms of ν and n_B

$$|\frac{d\Delta_n}{d\Delta}|_{\Delta_c} \sim N^{1/\nu}. \quad (28)$$

The comparison with Eq.(25) demonstrates that the exponent which governs the nonanalytic behavior of the entanglement entropy and concurrence in the vicinity of the critical point is nothing than the inverse of correlation length exponent. It should be noted that the scaling of the position of minimum, Δ_m (Fig.5), also comes from the behavior of the correlation length near the critical point. As the critical point is approached and in the limit of large system sizes, not the thermodynamic limit, the correlation length almost covers the whole of system, i.e. $\xi \sim N$, and a simple comparison with Eq.(26) results in the scaling of position as $\Delta_m = \Delta_c + N^{-1/\nu}$.

V. FURTHER INSIGHT FROM QUANTUM GROUP

As it has been carried out in Sec.(II, III) the (block) quantum renormalization group is conceptually and technically simple, but it may yield poor quantitative results. This was the main reason on the slow development of QRG through the 1980s in favor of powerful numerical Quantum Mont Carlo method. However there was a comeback to the RG in the 1990s as one of the most powerful method when dealing with the zero temperature properties of the quantum systems. It was Wilson the first to associate the failure of the QRG to the role of the boundary conditions when applying to the tight bonding model. In fact the success of the density matrix renormalization group (DMRG) as developed by White³¹ refers to the way in which it takes into account the effect of bound-

ary conditions in terms of the quantum correlation in the ground state of the system. Later, it was shown that the breakdown of the QRG is rooted on taking into account the entanglement in the ground state of the system³².

However, Martin-Delgado, *et al*²⁰ proposed a QRG prescription which implements the concept of the quantum group to the renormalization group approach. In this approach the effect of boundary conditions have been imposed in terms of boundary magnetic fields on each block. The boundary fields cancel each other when collecting all blocks into the whole chain. This method was used to describe the critical line of the XXZ model for $0 \leq \Delta \leq 1$ and its basic idea relies on the "restoring" the rotational symmetry of the model by adding the appropriate boundary terms to the model. The open chain Hamiltonian is defined as

$$H = \frac{J}{4} \sum_{i=1}^N h_{i,i+1} \quad , \quad h_{i,i+1} = \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \left(\frac{q + q^{-1}}{2} \right) \sigma_i^z \sigma_{i+1}^z - \left(\frac{q - q^{-1}}{2} \right) (\sigma_i^z - \sigma_{i+1}^z) \right] \quad ,$$

Where q is an arbitrary complex number.

To construct a real space RG, we have considered three-site block (Fig.1) which is important in order to get

a renormalized Hamiltonian of the same form as the original one. In this case the inter-block (H^{BB}) and intra-block (H^B) Hamiltonians are:

$$h_I^B = \frac{J}{4} \left[(\sigma_{1,I}^x \sigma_{2,I}^x + \sigma_{2,I}^x \sigma_{3,I}^x + \sigma_{1,I}^y \sigma_{2,I}^y + \sigma_{2,I}^y \sigma_{3,I}^y) + \left(\frac{q + q^{-1}}{2} \right) (\sigma_{1,I}^z \sigma_{2,I}^z + \sigma_{2,I}^z \sigma_{3,I}^z) - \left(\frac{q - q^{-1}}{2} \right) (\sigma_{1,I}^z - \sigma_{3,I}^z) \right],$$

$$H^{BB} = \frac{J}{4} \sum_{I=1}^{N/3} \left[(\sigma_{3,I}^x \sigma_{1,I+1}^x + \sigma_{3,I}^y \sigma_{1,I+1}^y) + \left(\frac{q + q^{-1}}{2} \right) (\sigma_{3,I}^z \sigma_{1,I+1}^z) - \left(\frac{q - q^{-1}}{2} \right) (\sigma_{3,I}^z - \sigma_{1,I+1}^z) \right].$$

In this way the correlations between blocks are taken into account in the coarse graining procedure in terms of

the boundary fields of each block. The ground state of the block Hamiltonian is doubly degenerate and given by

$$|\psi_0\rangle = \frac{1}{\sqrt{2(q + q^{-1} + 1)}} (-q^{1/2} |\uparrow\uparrow\downarrow\rangle + (q^{1/2} + q^{-1/2}) |\uparrow\downarrow\uparrow\rangle - q^{-1/2} |\downarrow\uparrow\uparrow\rangle),$$

$$|\psi'_0\rangle = \frac{1}{\sqrt{2(q + q^{-1} + 1)}} (-q^{1/2} |\uparrow\downarrow\downarrow\rangle + (q^{1/2} + q^{-1/2}) |\downarrow\uparrow\downarrow\rangle - q^{-1/2} |\downarrow\downarrow\uparrow\rangle).$$

and the corresponding energy is $e_0 = -\frac{J}{4}(2 + q + q^{-1})$.

The effective Hamiltonian with renormalized couplings is

$$H^{eff} = \frac{J'}{4} \left[\sum_i^{N/3} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \left(\frac{q' + q'^{-1}}{2} \right) (\sigma_i^z \sigma_{i+1}^z) - \left(\frac{q' - q'^{-1}}{2} \right) (\sigma_i^z - \sigma_{i+1}^z) \right],$$

where

$$q' = q, \quad J' = \xi^2(q)J, \quad \xi(q) = \frac{q + q^{-1} + 2}{2(q + q^{-1} + 1)} \quad (29)$$

The remarkable result of eq.(29) is that the coupling constant q when set as a pure phase, or alternatively $\Delta = (\frac{q+q^{-1}}{2})$ dose not flow under RG transformation which predicts correctly a line of critical models in the range $|\Delta| < 1$, while J^m , which is the value of J after m RG iterations, goes to zero in the limit where $m \rightarrow \infty$, which in turn defines the scale of energy. This means that in the critical region the entanglement measure, either concurrence or entanglement entropy do not evolve through the renormalization of the anisotropic coupling constant, i.e. it does not rescale the anisotropy parameter ($\Delta' = \Delta$) since every point in this region is a fixed point.

The density matrix is constructed from either $|\psi_0\rangle$ or $|\psi'_0\rangle$. We then sum over site 1 and 3 degrees of freedom to get the reduced density matrix of site 2 and the rest of system. The von-Neumann entropy or the entanglement of site 2 and the whole of system is obtained by the eigenvalues of the reduced density matrix (similar to Eq.(24)). The entanglement of site 2 (E_q) has been plotted in Fig.7 versus the anisotropy parameter. For $0 < \Delta \leq 1$ we observe a decrease of the entanglement versus Δ which is the effect of anisotropy to reduce the quantum correlations whilst it is maximum at $\Delta = 0$. Moreover, the reported values for $0 < \Delta \leq 1$ do not evolve under RG transformation which shows to be the value as $N \rightarrow \infty$. For $\Delta > 1$ we get the evolution of E_q in terms of RG transformation. For higher RG iterations we get the zero value of entanglement which manifests the uncorrelated nature of a (classical) Ising state. The nonanalytic behavior which manifest itself in the first derivative of the entanglement comes as the critical point $\Delta_c = 1$ is approached from the Ising (gapped) phase and is accompanied by a scaling with an exponents like what were seen before.

VI. CONCLUSIONS

In this work the relation between the entanglement and quantum phase transition via the renormalization group procedure is addressed. We have used the idea of renormalization group to study the quantum information properties of the XXZ chain. The entanglement entropy and concurrence as two measures of quantum correlations are used. In order to explore the critical behavior

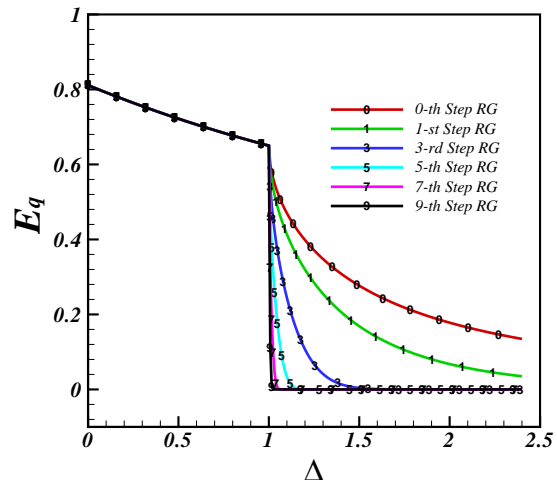


FIG. 7: (color online) Quantum group manifestation of the evolution of Entanglement entropy at different RG iterations (steps).

of the XXZ model the evolution of both entanglement entropy and concurrence through the renormalization of the lattice were examined. As the number of RG iterations increases the entanglement entropy as well as concurrence develops two different values in both sides of the quantum critical point, where the phase transition between the spin fluid phase and the Ising like phase occurs. The phase transition becomes significant which shows a diverging behavior in the first derivative of the two measures. This divergence of entanglement measures accompanied by some scaling behavior near the critical point (as the size of the system becomes large). The scaling behavior characterizes how the critical point of the model is touched as the system size is increased. The fact that both measures of entanglement scale with the same exponent is a significant result of the large scale behavior of the model near the critical point. It is also shown that the nonanalytic behavior of both measures of entanglement is mirrored from the correlation length exponent in the vicinity of the critical point. This inspires the behavior of the entanglement near the critical point is directly connected to the quantum critical properties of the model.

Compensating the poor results of the QRG in the critical region of the XXZ model, we resort to the quantum group method in order to take into account the corre-

lation between blocks. In this way the massless region is truly captured and a realistic feature of the evolution of the concurrence or entanglement entropy is inspired. However, the nonanalytic behavior and the scaling behavior is the same as before as the critical point of the model is approached from the gapped phase.

The approach presented here can be used to get the quantum information properties of quantum systems in an easy way. In other words we get the properties of a large system dealing with a small block which make it possible to get analytic results. However, the quantum renormalization group usually suffers from the poor quantitative results. One should get more accurate results using the proposed idea with a more complex

numerical method like density matrix renormalization group or exact diagonalization method.

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References

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- ¹ J. S. Bell, *Physics* **1**, 195 (1964).
 - ² M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Communication* (Cambridge University Press, Cambridge, 2000).
 - ³ S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 2000).
 - ⁴ A. Osterloh, Luigi Amico, G. Falci and Rosario Fazio, *Nature* **416**, 608 (2002).
 - ⁵ L.A. Wu, M. S. Sarandy, and D. A. Lidar, *Phys. Rev. Lett.* **93** 250404 (2004).
 - ⁶ J. I. Latorre, C. A. Lütken, E. Rico and G. Vidal, *Phys. Rev. A* **71**, 034301 (2005).
 - ⁷ G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, *Phys. Rev. Lett.* **90**, 227902 (2003).
 - ⁸ J. Vidal, G. Palacios, and R. Mosseri, *Phys. Rev. A* **69**, 022107 (2004).
 - ⁹ T. J. Osborne and M. A. Nielsen, *Phys. Rev. A* **66**, 032110 (2002).
 - ¹⁰ I. Bose and E. Chattopadhyay, *Phys. Rev. A* **66**, 062320 (2002).
 - ¹¹ F. Verstraete, M. Popp, and J. I. Cirac, *Phys. Rev. Lett.* **92**, 027901 (2004).
 - ¹² P. Zanardi and X. Wang, *J. Phys. A* **35**, 7947 (2002)
 - ¹³ Shi-Jian Gu, Shu-Sa Deng, You-Quan Li, and Hai-Qing Lin, *Phys. Rev. Lett.* **93**, 086402 (2004).
 - ¹⁴ Alberto Anfossi, Paolo Giorda, and Arianna Montorsi, *Phys. Rev. B* **75**, 165106 (2007)
 - ¹⁵ M. Kargarian, R. Jafari and A. Langari, *Phys. Rev. A* **76**, 60304 (R) (2007).
 - ¹⁶ F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf *Phys. Rev. Lett.* **94**, 140601 (2005).
 - ¹⁷ P. Pfeuty, R. Jullian, K.L. Penson in: *Real-Space Renormalization*, eds. T.W. Burkhardt, J.M.J. van Leeuwen (Springer, Berlin, 1982) ch. 5.
 - ¹⁸ K. G. Wilson, *Rev. Mod. Phys.* **47**, 773 (1975).
 - ¹⁹ Shi-Jian Gu, Hai-Qing Lin, and You-Quan Li, *Phys. Rev. A* **68**, 042330 (2003)
 - ²⁰ M. A. Delgado, G. Sierra *Phys. Rev. Lett.* **76**, 1146 (1996)
 - ²¹ M. A. Martin-Delgado and G. Sierra, *Int. J. Mod. Phys. A* **11**, 3145 (1996).
 - ²² A. Langari, *Phys. Rev. B* **69**, 100402(R) (2004); A. Langari, *Phys. Rev. B* **58**, 14467 (1998).
 - ²³ J. I. Latorre, E. Rico, and G. Vidal, *Quant. Inf. Comput.* **4**, 48 (2004)
 - ²⁴ R. Jafari and A. Langari, *Phys. Rev. B* **76**, 014412 (2007); R. Jafari and A. Langari, *Physica A* **364**, 213-222 (2006)
 - ²⁵ Shi-Jian Gu, Guang-Shan Tian, and Hai-Qing Lin, *Phys. Rev. A* **71**, 052322 (2005)
 - ²⁶ Shi-Jian Gu, Guang-Shan Tian, and Hai-Qing Lin, *New Journal of Physics* **8**, 61 (2006)
 - ²⁷ W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998). C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A*, **54**, 3824 (1996).
 - ²⁸ V. Coffman, J. Kundu and W.K. Wootters *Phys. Rev. A* **61**, 052306 (2000)
 - ²⁹ V. E. Korepin, *Phys. Rev. Lett.* **92**, 096402 (2004)
 - ³⁰ Y. Chen, P. Zanardi, Z. D. Wang, and F. C. Zhang, *New Journal of Physics* **8**, 97 (2006), quant-ph/0407228
 - ³¹ S.R. White, R.M. Noack, *Phys. Rev. Lett.* **68**, 3487 (1992)
 - ³² T. J. Osborne and M. A. Nielsen, *Quantum Information Processing* **1**, 45 (2002). quant-ph/0109024